Class 21: Maps

Goals for the day:

1. Use map ideas to examine Newton's method of finding roots.
2. Become more familiar with the flip bifurcation and distinguish between sub-/super-critical cases.
3. Learn about the shift map (time permitting).

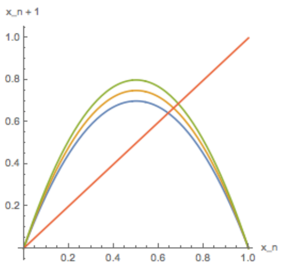
1. (10.1.12-13) A fixed point, x^*, is called *superstable* if $\lambda = \frac{df}{dx}|_{x^*} = 0$. To determine stability we check whether $|\lambda| < 1$, so $|\lambda| = 0$ is the minimum value it can attain. Usually for a perturbation η_0 away from the fixed point, $\eta_1 \sim \lambda\eta_0$ and $\eta_n \sim \lambda^n\eta_0$. For a superstable fixed point, convergence is much faster.

   Newton's method is a method of finding the roots of an equation of the form $g(x) = 0$. With this method, you make an initial guess of the root, x_0. You then update your guess in the following way: find the slope of $g(x)$ at your guess; starting at $(x_0, g(x_0))$ and moving along a line with slope $g'(x_0)$, determine where you will intersect the x-axis; use this intersection point as your new guess of the root.

   (a) Write a map $x_{n+1} = f(x_n)$ that encodes this algorithm in terms of x_n, $g(x_n)$ and $\frac{dg}{dx}|_{x_n}$.
   (b) Consider the example of $g(x) = x^2 - 4 = 0$. Write the "Newton map" for this example.
   (c) Find the fixed points of this map.
   (d) For the example map above, show that the fixed points are superstable.
   (e) Iterate the map starting from $x_0 = 1$. Take just two iterates. How close are you to a fixed point?
   (f) Now find an expression for the fixed points for general $g(x)$, assuming $g'(x) \neq 0$.
   (g) Show that these fixed points (of the general map) are superstable (assuming $g'(x) \neq 0$).

2. Consider the logistic map, given by $x_{n+1} = f(x_n)$ where $f(x) = rx(1 - x)$.

   (a) For what r does it have a superstable fixed point?
   (b) In the figure below, the logistic map is shown for three values of r (on either side of the flip bifurcation). The *flip bifurcation* happens when $f'(x^*) = -1$. A flip bifurcation is a point where period-doubling occurs. The bifurcation can be either supercritical or subcritical. To determine which it is, we need to analyze the stability of the two-cycle after the bifurcation. Let p and q be fixed points of the map $f^2(x) = f(f(x))$ that are not also fixed points of $f(x)$.



   Note that $q = f(p)$ and $p = f(q)$. Computing $\lambda = \frac{d}{dx}f(f(x))$, we find $\lambda = f'(f(p))f'(p) = f'(q)f'(p)$. If $|\lambda| < 1$ then the 2-cycle is stable. Determine the range of r for which this two-cycle is stable.

   The roots, p and q, are roots of the quartic given by $f(f(x))$. We know $x = 0$ and $x = 1 - \frac{1}{r}$ are also solutions to this quartic, so we can find p, q as roots of the remaining quadratic. They are $p, q = \frac{1}{2} + \frac{1}{2r} \pm \frac{\sqrt{(r-3)(r+1)}}{2r}$. In addition, $p + q = 1 + \frac{1}{r}$ and $pq = \frac{1+r}{r^2}$.

3. (10.3.7) If time permits, consider the decimal shift map, $x_{n+1} = 10x_n \mod 1$.

(a) Draw a map of this graph, restricting yourself to $0 \le x \le 1$.

(b) Identify the fixed points, using a decimal representation. Looking to your map, are they stable?

(c) Show the map has a periodic point of any period (by giving an explicitly constructed example, perhaps).

(d) Does the map have aperiodic orbits? Find an example of one.

(e) This map has sensitive dependence on initial conditions. Consider two nearby initial points and find their rate of separation.

Some Answers:

1. (a) We have $y = mx + b$ is the line through $(x_n, g(x_n))$ with slope $g'(x_n)$. So $m = g'(x_n)$. We want the intersection with the x-axis, so $0 = mx_{n+1} + b$. Since $g(x_n) = g'(x_n)x_n + b$, we know $b = g(x_n) - g'(x_n)x_n$. Given that $x_{n+1} = -\frac{b}{m}$, $\Rightarrow x_{n+1} = -\frac{g(x_n)-g'(x_n)x_n}{g'(x_n)}$. Simplifying, this becomes $x_{n+1} = f(x_n)$ where

$$f(x_n) = x_n - \frac{1}{\frac{dg}{dx}|_{x_n}} g(x_n).$$

   (b) $x_{n+1} = x_n - \frac{x_n^2-4}{2x_n} = \frac{x_n^2+4}{2x_n}$.

   (c) Fixed points are given by $x - \frac{x^2-4}{2x} = x$ and the xs cancel so $\frac{x^2-4}{2x} = 0$. This is zero when the numerator is zero, so when $x^2 - 4 = 0$. Thus $x = \pm 2$.

   (d) For stability, need to check $f'(x)$ which is $1 - \frac{2x}{2x} + \frac{x^2-4}{4x^2}2 = \frac{x^2-4}{2x^2}$ and this is 0 at $x = \pm 2$ so the fixed points are superstable.

   (e) $x_0 = 1$ so $x_1 = 1 - \frac{1-4}{2} = 1 + \frac{3}{2} = \frac{5}{2} = 2.5$. Then $x_2 = \frac{5}{2} - \frac{\frac{25}{4}-4}{5} = \frac{5}{2} - \frac{9}{20} = \frac{41}{20} = 2.05$. That was pretty fast convergence towards 2!

   (f) For general fixed points, $x - \frac{g}{g'} = x$ so $-\frac{g}{g'} = 0$. Away from $g' = 0$ this just means we need $g = 0$, and the fixed points are the roots.

   (g) For stability, $1 - \frac{g'}{g'} + \frac{g}{g'^2}g''$ and this simplifies to $\frac{g}{g'^2}g''$ but at the roots of $g = 0$, this is certainly 0 (so long as we're staying away from $g' = 0$), so the fixed points are superstable.

2. (a) f' at $1 - \frac{1}{r}$ is $2 - r$ so superstable when $r = 2$.

   (b) $\lambda = f'(q)f'(p) = r(1-2q)r(1-2p) = r^2[1 - 2(p+q) + 4pq] = 4 + 2r + r^2$. We want $|\lambda| < 1$. So $4 + 2r - r^2 = 1$ and $4 + 2r - r^2 = -1$ form the bounds in r. This means $r^2 - 2r - 3 = 0$ so $r = 3$ or $r = -1$ and $r^2 - 2r + 5 = 0$ so $r = 1 \pm \sqrt{6}$. $1 + \sqrt{6} \approx 3.449$ so $3 < r < 1 + \sqrt{6}$. So $3 < r < 1 + \sqrt{6}$.

3. Map is below. Fixed points are of the form 0.11111..., 0.2222..., etc. The slope of $f(x)$ is greater than 1 so the fixed points are unstable. To get period-2, 0.1212..., to get period-3 0.122122..., etc. For an aperiodic orbit, try $\sqrt{2} - 1$ or $\pi - 3$ or $e - 2$. For sensitive dependence, consider 0.111... and 0.1112. These start at $\delta_0 \approx 0.0001$ apart, then become 0.111... and 0.112 after one iterate so $\delta_1 \approx 0.001$ apart, then become 0.111... and 0.12 so $\delta_2 \approx 0.01$ apart. This is changing by a factor of 10 each time, so $\|\delta_n\| \sim \|\delta_0\|10^n = \|\delta_0\|e^{nln10}$. We call $\ln 10$ the Liapunov exponent in this map context.