AM147 11:30am

Class 26: 2D Maps

Goals for the day:

1. Explore some properties of the Hnon map.
2. Work with the Smale horseshoe.

Problems:

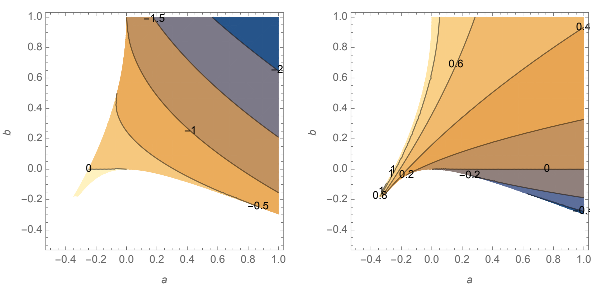
1. The Hénon map is given by

$$
\begin{aligned}
x_{n+1} &= 1 + y_n - a x_n^2 \\
y_{n+1} &= b x_n.
\end{aligned}
$$

   (a) (12.2.4) Find all of the fixed points of this map and give an existence condition for them.

   (b) (12.2.5) Calculate the Jacobian matrix of the Hnon map and find its eigenvalues.

   (c) (12.2.6) A fixed point is linearly stable if all eigenvalues satisfy $|\lambda| < 1$. Consider $-1 < b < 1$. The fixed points are of the form $x = -c \pm \sqrt{c^2 + d}$. The $x = -c - \sqrt{c^2 + d}$ fixed point is always unstable. Consider the $x = -c + \sqrt{c^2 + d}$ fixed point. Using the contour plots below for the value of each eigenvalue, what is its stability?



   (d) *Come back to this if time permits, but skip ahead to the Smale horseshoe for now.* Along the line $a = \frac{3}{4}(1-b)^2$, one of the eigenvalues attains $\lambda = -1$. Show that for $a > \frac{3}{4}(1-b)^2$, there is a 2-cycle.

2. (12.1.7) The Smale horseshoe map is illustrated in the figure below. In this map, some of the points that start in the unit square are mapped outside the square after an iteration of the map.

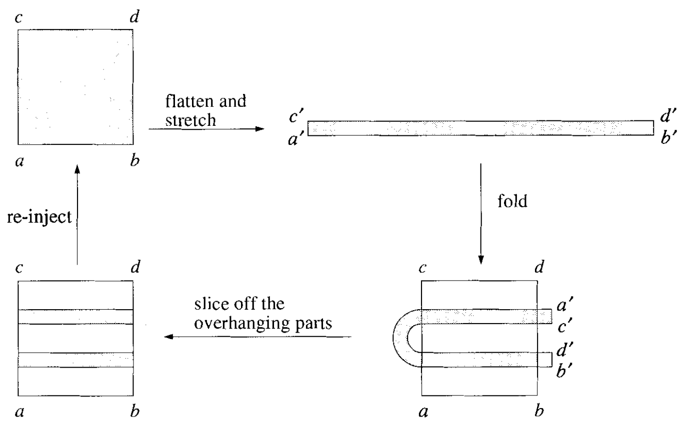


Figure 1: The Smale horseshoe map (from Strogatz)

(a) In the original unit square, which regions remain in the unit square after one iteration? Mark these regions V_0 and V_1.

(b) Sketch the effect of a second iteration of the map. Identify the points in the original unit square that survived two iterations. Mark these regions $V_{00}, V_{01}, V_{10}, V_{11}$.

(c) Work to identify the set of points in the original unit square that survive forever under forward iterations of the map.

(d) Now consider a backward iterate of the map. Which points stay in the unit square under a backward iteration? Mark these regions H_0 and H_1

(e) What about under two backward iterations? Mark these regions H_{00}, etc.

(f) Attempt to construct the set of points that is in the unit square for all time (both forward and backward).

Some answers:

1. (a) $x^* = \frac{-1+b}{2a} \pm \sqrt{\left(\frac{-1+b}{2a}\right)^2 + 1}$, $y^* = bx^*$, $\left(\frac{-1+b}{2a}\right)^2 + 1 > 0$.

   (b) $\lambda = -ax^* \pm \sqrt{(ax^*)^2 + b}$

   (c) Stable until λ_1 crosses -1. Then there is a flip bifurcation.