Physics 15c: Dispersion Review

October 21, 2013

1 Dispersion

A dispersion relation is the functional dependence of the frequency ω of a wave on its wavenumber k:

Dispersion relation: $\omega(k)$.

The dispersion relation for a given wave equation is found by plugging the normal mode solution $\psi(x,t) = Ae^{i(kx-\omega t)}$ into the wave equation and solving for ω as a function of k.

Two important velocities related to wave propagation can be derived from the dispersion relation.

(a) The *phase velocity* describes the velocity of a traveling wave with a given frequency and wavelength.

Phase velocity$\equiv c \equiv \frac{\omega}{k}$

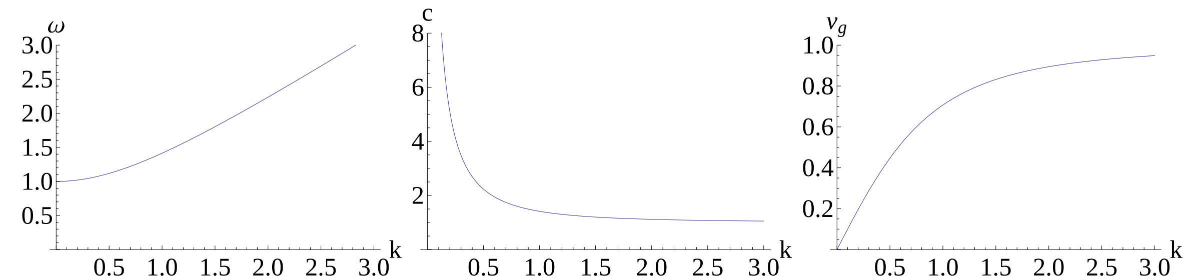
(b) The *group velocity* describes the velocity of a pulse envelope that made up of a narrow range of frequencies around ω.

Group velocity$\equiv v_g \equiv \frac{d\omega}{dk}$

For the following dispersion relation, sketch a plot of (a) the dispersion relation, $\omega(k)$ vs. k; (b) the phase velocity vs. k; and (c) the group velocity vs. k. Also, in (a), choose a point on the curve $\omega(k)$ and illustrate graphically the phase and group velocities at that point.

$$\omega(k) = \sqrt{k^2+1} \tag{1}$$

The answers are on the next page.



Dispersion relations can either be *non-dispersive* or *dispersive*:

Non-dispersive: $\omega = ck$.

This is the dispersion relation for the ordinary wave equation $\frac{\partial^2\psi}{\partial t^2} = c^2\frac{\partial^2\psi}{\partial x^2}$, for example. *Non-dispersiveness* $\Longrightarrow$ the shape of a propagating waveform is unchanged.

Draw below an example of a waveform propagating non-dispersively.

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Dispersive: $\omega(k) \neq ck$.

Dispersiveness $\Longrightarrow$ the shape of a propagating waveform breaks apart because the different frequencies that make up the waveform are propagating at different velocities.

Draw below an example of a waveform propagating dispersively.

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2 Beaded String

The wave equation for a transverse wave on a beaded string of tension T with beads of mass m and separation ℓ is derived in Morin 6.1:

$$\frac{\partial^2 \psi_n}{\partial t^2} = \frac{T}{m\ell}(\psi_{n+1} - 2\psi_n + \psi_{n-1}). \tag{2}$$

where ψ is the transverse displacement of the string at bead number n and time t. For convenience, we can define the quantity

$$\omega_0^2 = \frac{T}{m\ell}. \tag{3}$$

The dispersion relation for this system is [Morin 6.1, Eq. (9)]

$$\omega(k) = 2\omega_0 \sin\left(\frac{k\ell}{2}\right). \tag{4}$$

From Eq. (4), we know that the phase velocity is

$$c(k) \equiv \frac{\omega}{k} = \frac{2\omega_0 \sin(k\ell/2)}{k}. \tag{5}$$

The group velocity is

$$v_g \equiv \frac{d\omega}{dk} = \omega_0 \ell \cos\left(\frac{k\ell}{2}\right). \tag{6}$$

This system displays a *high-frequency cutoff* in that waves with frequency $\omega > 2\omega_0$ cannot propagate in it [see Eq. (4)]. If an end of the string is driven above this frequency, the waves that are generated undergo total reflection after entering the medium with exponentially decaying amplitude. The solution to the wave equation with $\omega > 2\omega_0$ is called an *evanescent* wave, and is has the form:

$$\psi(x,t) = Ae^{-\kappa n\ell}(-1)^n \cos(\omega t + \phi), \tag{7}$$

where $i\kappa + \pi/\ell = k$.

3 Spring-Loaded String

Consider a continuous string connected to a rigid ceiling by many springs along the length of the spring. The wave equation for a transverse wave on this string with tension T, linear mass density μ, and spring constant per unit length σ is derived in Morin 6.2.2:

$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2} - \omega_s^2 \psi. \tag{8}$$

where ψ is the transverse displacement of the string as a function of position x and time t, $c = \sqrt{T/\mu}$, and $\omega_s = \sigma/\mu$.

The dispersion relation for this system is [Morin 6.2.2, Eq. (19)]

$$\omega(k) = \sqrt{c^2 k^2 + \omega_s^2}. \tag{9}$$

From Eq. (9), we know that the phase velocity is

$$c(k) \equiv \frac{\omega}{k} = \sqrt{c^2 + \omega_s^2/k^2}. \tag{10}$$

The group velocity is

$$v_g \equiv \frac{d\omega}{dk} = \frac{c^2 k}{\sqrt{c^2 k^2 + \omega_s^2}}. \tag{11}$$

This system displays a *low-frequency cutoff* in that waves with frequency $\omega < \omega_s$ cannot propagate in it [see Eq. (9)]. If an end of the string is driven below this frequency, the waves that are generated undergo total reflection after entering the medium with exponentially decaying amplitude. The solution to the wave equation with $\omega < \omega_s$ is called an *evanescent* wave, and is has the form:

$$\psi(x,t) = Be^{-\kappa x}\cos(\omega t + \phi), \tag{12}$$

where $i\kappa = k$.