Class 24: Fractals

Goals for the day:

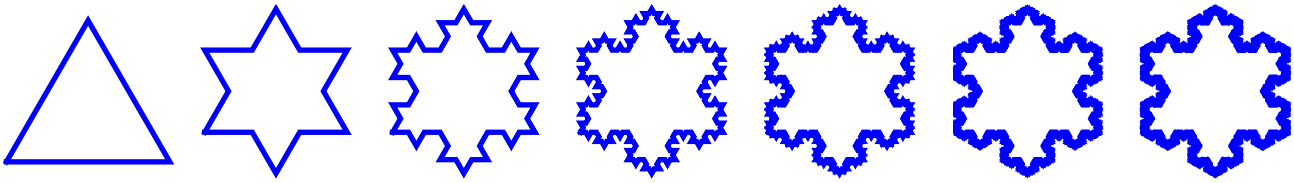
1. Use similarity dimension to determine fractal dimensions.

Problems:

1. (11.1.1) Consider the diagonal argument used to show that $S = \{x : 0 \leq x < 1\}$ is an uncountable set. Why doesn't this argument also show that the rational numbers are uncountable?

2. (11.2.1) We want to find the total length of the points in the Cantor set. To do this, consider the lengths of the intervals that we removed to construct the set. First we removed an interval of length $\frac{1}{3}$. Then we removed two intervals, each of length $\frac{1}{9}$, etc. Show that the total length of all of the removed intervals is 1 and this the leftover points making up the Cantor set have length 0.

3. (11.3.7) The von Koch snowflake curve is shown below. It is made by starting with an equilateral triangle and then replacing each side using the von Koch procedure.

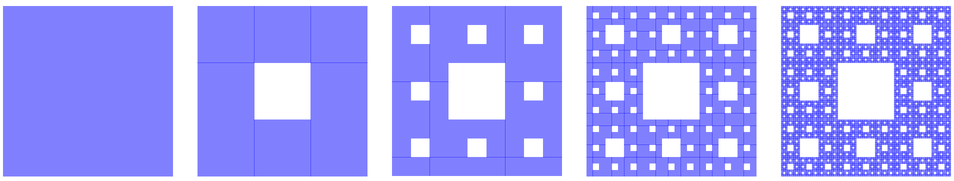


   (a) Show that the snowflake curve has infinite arc length.

   (b) Assuming that the initial side lengths are 1, find the area of the region enclosed by the snowflake.

   (c) Find the similarity dimension of the snowflake. *Recall that # copies = (scale)*^{d}.

4. (11.3.8) The Sierpinski carpet is formed by dividing a closed unit box into nine equal boxes, removing the central (open) box, and repeating.



   (a) Find the similarity dimension.

   (b) Show the Sierpinski carpet has zero area.

5. (11.3.10) A fat fractal has nonzero measure (length for $d < 1$, area for $1 < d < 2$). Consider the set created by starting with $[0, 1]$, deleting the open middle $\frac{1}{2}$, then the open middle $\frac{1}{4}$ of the two remaining subintervals, then the open middle $\frac{1}{8}$, etc.

   (a) Show that the length of the limiting set is greater than zero.

   (b) Show that the set is a topological Cantor set. First show it is "totally disconnected". To do this, pick any two points in the set, and argue that they are in different "halves" of the set at some point. Then show that it has no "isolated points". To do this, show that for any point in the set, p, and any neighborhood length, ϵ, there is a second point within distance ϵ of p.

Some Answers:

1. We make a list of all of the rational numbers with their decimal expansions. As we construct a new number that is not on the list (and this is definitely something we can do), it is not clear that the new number is rational - it definitely does not have a finite decimal expansion and there is no reason the decimals would be repeating.

2. We remove $L = \frac{1}{3} + 2\frac{1}{9} + 4\frac{1}{27} + ... = \sum_1^\infty \frac{2^{n-1}}{3^n}$. So $\frac{2}{3}L = \frac{2}{9} + \frac{4}{27} + ...$ Thus $L - \frac{2}{3}L = \frac{1}{3}$. This means $\frac{1}{3}L = \frac{1}{3}$, so $L = 1$.

3. The number of line segments is a sequence $3, 4*3, 4*4*3, ..., 4^k 3, ...$ so $n_k = 4^k 3,\ k = 0, 1, 2, ...$ Length of a line segment at step k gives a sequence $1, \frac{1}{3}, \frac{1}{9}, ..., \frac{1}{3^k}, ...,$ so $l_k = \frac{1}{3^k},\ k = 0, 1, 2, ...$ Total length at each iterate is $n_k l_k$, so $3, 4, 4\frac{4}{3}, ..., 4\frac{4^{k-1}}{3^{k-1}}, ...$ In the limit $k \to \infty$, this is unbounded.

   For the area, we start with area A_0 in the triangle. Then the next structure adds n_0 triangles each with area $\frac{1}{9}$ the original area. Area sequence is

$$A_0, A_0 + 3\frac{1}{9}A_0, A_0 + \frac{1}{3}A_0 + 4*3\frac{1}{3^4}A_0, ..., A_0 + \sum_{r=1}^{k} n_{r-1}{l_r}^2 A_0, ...$$

   So $a_k = A_0(1 + \sum_{r=1}^k 4^{r-1} 3 \frac{1}{3^{2r}}) = A_0(1 + \frac{1}{3}\sum_{r=0}^{k-1} (\frac{4}{9})^r)$. The area of the fractal is $A = A_0(1 + \frac{1}{3}\sum_{r=0}^{\infty} (\frac{4}{9})^r)$. $\frac{4}{9}A = A_0(\frac{4}{9} + \frac{1}{3}\sum_{r=1}^{\infty}(\frac{4}{9})^r)$, so $A - \frac{4}{9}A = A_0(\frac{5}{9} + \frac{1}{3})$. Thus $A = \frac{9}{5}\frac{8}{9}A_0 = \frac{8}{5}A_0$.

   Still scaling down by 3 and making 4 copies, so $d = \frac{\ln 4}{\ln 3} \approx 1.26$.