Class 25: 2D Maps

Goals for the day:

1. Determine when the origin is stable for a linear map.
2. Show that the Hénon map is not area preserving via composition of transformations.
3. Explore the Baker's map.

Problems:

1. (12.1.1) Consider the 2D linear map

$$x_{n+1} = ax_n$$
$$y_{n+1} = by_n,$$

$a, b \in \mathbb{R}$. Identify the possible patterns of orbits near the origin depending on the signs and sizes of a and b. Draw the possible patterns for orbits that converge to the origin. *Note that the map is linear and uncoupled.*

2. (12.1.2) Now consider the 2D linear map

$$x_{n+1} = ax_n + by_n$$
$$y_{n+1} = cx_n + dy_n,$$

$a, b, c, d \in \mathbb{R}$. Find conditions on the parameters so that the origin is globally attracting (globally asymptotically stable).

A linear algebra interlude: this equation can be written $\mathbf{x}_{n+1} = A\mathbf{x}_n$. Any matrix A can be transformed in the following way: $A = P^{-1}DP$ where D is diagonal or $A = P^{-1}CP$ where C is an upper triangular matrix and the diagonal entries of C or D are the eigenvalues of A. Here, C or D is known as the ***Jordan normal form*** *of the matrix, and P, D, C may be over the complex numbers. Recall the following: $P^{-1}P = PP^{-1} = \mathbb{I}$; if $D = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ then $D^k = \begin{pmatrix} \alpha^k & 0 \\ 0 & \beta^k \end{pmatrix}$; if $C = \begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix}$ then $C^k = \begin{pmatrix} \alpha^k & \gamma_k \\ 0 & \beta^k \end{pmatrix}$ for some $\gamma_k \in \mathbb{C}$.*

3. The Hénon map is given by $x_{n+1} = 1 + y_n - ax_n^2$ and $y_{n+1} = bx_n$. Consider the series of transformations $T' : x' = x, y' = 1 + y - ax^2$, $T'' : x'' = bx', y'' = y'$, $T''' : x''' = y'', y''' = x''$.

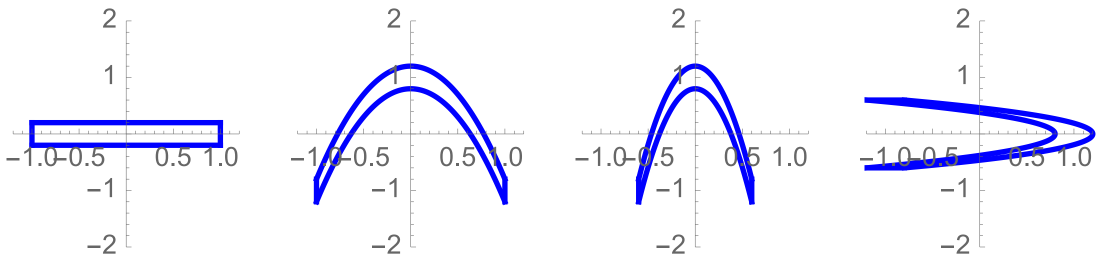

Figure 1: The transformations T', T'' and T''' are composed from left to right, with T' operating on the rectangle on the far left.

   (a) (12.2.1) Show that composing this series ($T'''T''T'$) of transformations yields the Hénon map.

(b) (12.2.2) Show that the transformations T' and T'' are area preserving but T'' is not.

A vector calculus interlude: think of the map T' as a coordinate transformation from coordinates xy to coordinates $x'y'$. We are interested in the area of a region of the xy plane after it undergoes the coordinate transformation. Recall: $\iint_R dx\ dy = \iint_S \left|\frac{\partial(x,y)}{\partial(x',y')}\right| dx'dy'$ where $\frac{\partial(x,y)}{\partial(x',y')} = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{vmatrix}$.

4. The Baker's map is given by

$$B(x_n, y_n) = (x_{n+1}, y_{n+1}) = \begin{cases} (2x_n, ay_n) & \text{for } 0 \le x_n \le \frac{1}{2} \\ (2x_n - 1, ay_n + \frac{1}{2}) & \text{for } \frac{1}{2} \le x_n \le 1 \end{cases}.$$

It is illustrated by Figure 12.1.4 of the text, shown below.

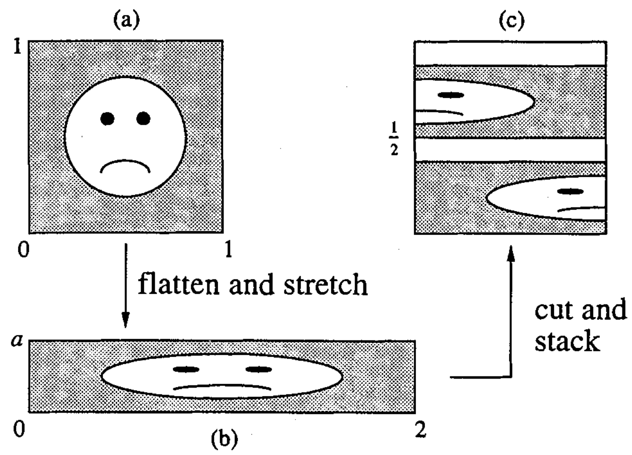

Figure 12.1.4

(a) Explain why the map is equivalent to the procedure of stretching by 2 and flattening by a, then cutting and stacking, that is shown in the figure.

(b) Sketch what will happen after one more iterate of the map shown in the figure. (Include the face!)

(c) This process should remind you of forming the Cantor set. Consider covering the n^{th} iterate of the map with square boxes of side length a^n. Note that the first iterate has 2 stripes and the second has 4. The box dimension is given by $d = \lim_{\epsilon \to 0} \frac{\ln N}{\ln \frac{1}{\epsilon}}$ where N is the number of boxes needed to cover the set and ϵ is the side length of the boxes. Compute the box dimension for the limiting set of the Baker's map.

(d) In the case $a = \frac{1}{2}$, your box dimension should be 2 because the map is area preserving. Check that this is the case.

(e) (12.1.5) For the area preserving Baker's map, consider a binary representation of a point in the unit square:

$$(x, y)_2 = (0.a_1a_2a_3..., 0.b_1b_2b_3...)$$

where $a_1 = 0$ indicates the point has $0 \le x < \frac{1}{2}$ and $a_1 = 1$ indicates the point has $\frac{1}{2} \le x < 1$. Find the binary representation of $B(x, y)$.

Multiplying a coordinate by 2 has the effect of shifting the decimal place once to the right.

(f) Represent the point (x, y) as $...b_3b_2b_1.a_1a_2a_3....$ In this notation, what is $B(x, y)$?

(g) Use the binary version of the map to show that B has a period-2 orbit. Plot the locations of the two points involved in the orbit in the unit square.

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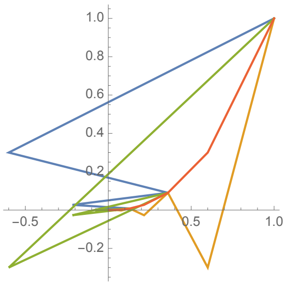
Some Answers:

1. The map is uncoupled, so we can tackle x and y separately. For x, if $|a| > 1$ then orbits diverge. If $|a| < 1$ then orbits converge. For $a > 0$, the change in x is monotonic. For $a < 0$, we alternate sides of the origin. All of the same is true for y, so there are sixteen possibilities. There are four possibilities that lead to convergence: $\{(a, b) : |a| < 1, |b| < 1\}$ is the interior of the unit square and the qualitative possibilities correspond to points in each of the four quadrants.



2. P and P^{-1} serve as an invertible pair of transformations. We can think about the system $P\mathbf{x}_{n+1} = CP\mathbf{x}_n$. If $P\mathbf{x_n}$ approaches the origin then $\mathbf{x}_n$ does as well. Let $\mathbf{z}_n = P\mathbf{x}_n$. $\mathbf{z}_0 = \begin{pmatrix} r \\ s \end{pmatrix}$. $\mathbf{z}_k = D^k\mathbf{z}_0 = \begin{pmatrix} \alpha^k r \\ \beta^k s \end{pmatrix}$. In this case it is clear that we need $|\alpha| < 1$ and $|\beta| < 1$ so we need both eigenvalues of the matrix to be less than 1.

   The nondiagonal case is slightly trickier. $\mathbf{z}_k = C^k\mathbf{z}_0 = \begin{pmatrix} \alpha^k r + c_k s \\ \beta^k s \end{pmatrix}$. It is not obvious how to deal with the $c_k s$ term. We can see it is not a problem by iterating another k times. $\mathbf{z}_{2k} = C^k C^k \mathbf{z}_0 = C^k \begin{pmatrix} \alpha^k r + c_k s \\ \beta^k s \end{pmatrix} = \begin{pmatrix} \alpha^k(\alpha^k r + c_k s) + c_k\beta^k s \\ \beta^{2k} s \end{pmatrix}$. In this case, in the limit as $k \to \infty$, we still need $|\alpha| < 1$ and $|\beta| < 1$ to approach the origin.

3. (a)

   (b) For T', $\begin{vmatrix} 1 & 0 \\ -2ax & 1 \end{vmatrix} = 1$. For T'', $\begin{vmatrix} b & 0 \\ 0 & 1 \end{vmatrix} = b$. For T''', $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = |-1| = 1$.

4. (a) For the unit square, consider the sets $S_0 = \{(x, y) : 0 \le x < \frac{1}{2}, 0 \le y < 1\}$ and $S_1 = \{(x, y) : \frac{1}{2} \le x < 1, 0 \le y < 1\}$. Under the action of the map, in the x direction, S_0 is stretched by a factor of two to take up the whole range $0 \le x < 1$ and y is squished by a factor of a. It is clear that this is the same thing as what happens to S_0 under a stretching by 2 and a flattening by a and then cutting, as S_0 is not impacted by the cutting and stacking procedure. For S_1, it is also stretched and flattened. Then $(\frac{1}{2}, 0)$ corner of S_1 is placed at $(0, \frac{1}{2})$, setting the placement of the whole stretched/flattened set. This is equivalent to what happens to the set under flattening/stretching and cutting/stacking.

   (b)

   (c) We have 2^n stripes and need $\frac{1}{a^n}$ boxes to cover a single stripe (stripes are of width a^n), so there are $\left(\frac{2}{a}\right)^n$ boxes being used and the box size is a^n. $d = \lim_{n\to\infty} \frac{\left(\frac{2}{a}\right)^n}{\ln \frac{1}{a^n}} = 1 - \frac{\ln 2}{\ln a}$

   (d) If we plug in $a = \frac{1}{2}$ we have $d = 1 - \frac{\ln 2}{-\ln 2} = 2$.

   (e) The x coordinate should be right shifted by the stretch, so it becomes $a_1.a_2a_3a_4....$ Cutting and stacking turns it into $0.a_2a_3a_4....$ For the y coordinate, it depends on the x coordinate. If $a_1 = 0$ then y becomes $0.0b_1b_2...$ while if $a_1 = 1$ then y becomes $0.1b_1b_2....$ So $(0.a_1a_2a_3, 0.b_1b_2b_3) \mapsto (0.a_2a_3..., 0.a_1b_1b_2...)$.

   (f) $...b_3b_2b_1.a_1a_2a_3... \mapsto ...b_2b_1a_1.a_2a_3a_4...$ so the map acts as a shift map on this representation.

(g) For a period-2 orbit, we are looking for a binary number that returns to itself after two shifts. These are the repeating fractions ...101010.101010... and ...010101.010101.... Their coordinates are given by $x = \frac{1}{2} + \frac{1}{8} + ..., y = \frac{1}{4} + \frac{1}{16} + ...$ and vice versa. Thus $x - \frac{1}{4}x = \frac{1}{2} \Rightarrow x_1 = \frac{2}{3}$ and $y - \frac{1}{4}y = \frac{1}{4} \Rightarrow y = \frac{1}{3}$. The points are $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{2}{3})$.