

MATHEMATICS 121, FALL 2012
LINEAR ALGEBRA WITH APPLICATIONS
Module #10 (Projective Geometry and Dual Spaces)

Last modified: October 8, 2012

Reading

- None. I learned the (fairly standard) proof of the theorem of Desargues from Ryan, *Euclidean and non-Euclidean geometry: an analytic approach*, but Ryan never mentions finite geometry.

Because the “points at infinity” are impossible to draw on a diagram, projective geometry over the real numbers is much harder to visualize than projective geometry over a finite field, although algebraically it is exactly the same. For that reason I advise against reading any of the standard books on the subject, although many of them are excellent.

Warmups(intended to be done before lecture)

- On page 4, each instructor and each committee of the small projective senate is represented by a vector. Convince yourself that the cross product of two instructors correctly specifies (up to a sign) the committee on which they serve together, and that the cross product of two committees correctly specifies (up to a sign) the instructor who serves on both.
- Redraw the diagram for the theorem of Desargues (page 9) so that P and P' are shown on opposite sides of X , as are Q and Q' and R and R' . Find L, M , and N by drawing lines and finding their intersections. If you are careful, your diagram will show that L, M , and N are collinear.

Lecture topics.

- 10.1 State and prove the theorem of Desargues in projective geometry. You may choose four points to have coordinates $(1, 1, 1)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ respectively.

Lecture topics.

1. Projective geometry as linear algebra

Although projective geometry developed out of the theory of perspective that was invented by Renaissance artists, it is most simply described in terms of vector spaces:

- Start with a three-dimensional abstract vector space V .
- The points of the projective geometry PV are the one-dimensional subspaces of V , each of which corresponds to a line through the origin.
- A point in PV can be represented by any vector in V that forms a basis for the subspace, but any nonzero multiple of this vector represents the same projective point.
- The lines of the projective geometry PV are the two-dimensional subspaces of V , each of which corresponds to a plane through the origin.
- A line in PV can be represented by any pair of vectors in V that form a basis for the subspace, but any independent pair of linear combinations of these vectors represents the same projective line.
- A line in PV is also the kernel of a linear function $f_\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$. The vector space of linear functions $f_\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a three-dimensional vector space V^* , called the dual space of V .
- If we introduce a basis into V and thereby identify it with \mathbb{R}^3 , then every nonzero column vector specifies a point in PV , while every nonzero 1×3 matrix α^T specifies a linear function f_α . Function f_α may be calculated by the rule $f_\alpha(\vec{v}) = \alpha \cdot \vec{v} = [\alpha^T][\vec{v}]$.
- $V = \mathbb{R}^3$ and $V^* = \mathbb{R}^3$ correspond to different abstract vector spaces. A sum or cross product needs two vectors from the same space; a dot product needs one vectors from each space.



2. How to make a projective senate from an affine senate of order n .

For each of the $n + 1$ pencils of parallel committees, add an "instructor at infinity" (also called an "ideal instructor" or an "exceptional instructor") to each committee in the pencil. Add a "committee at infinity" whose members are the $n + 1$ instructors at infinity. Now each committee has $n + 1$ members, and there are $n(n + 1) + 1$ committees and $n(n + 1) + 1$ instructors. In a listing of a projective senate, there is nothing special about the instructors or committees at infinity. Any asymmetry is only in the method of construction.

Now the geometric properties may be expressed as follows:

- Two distinct instructors determine a unique committee.
- Two distinct committees determine a unique instructor.

There is complete "duality" between committees and instructors.

We know how to assign to each instructor in an affine senate a pair of coordinates from the finite field F_n whose order equals the number of members n on a committee (n is a power of a prime.) Each of these instructors keeps its coordinates, and acquires a third coordinate equal to 1. This is just like

representing a point $\begin{pmatrix} x \\ y \end{pmatrix}$ of \mathbb{R}^2 by the vector $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, which we already did

when describing the isometries of Euclidean geometry. Each instructor at infinity is associated with a parallel pencil of lines, all of which have the same direction vector. If the lines in the pencil have "finite slope m ," take the direction vector as $\begin{pmatrix} 1 \\ m \end{pmatrix}$, otherwise as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (direction vector along the y axis). In either case the instructor at infinity gets a third component

of 0 and is represented by $\begin{bmatrix} 1 \\ m \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. In real projective geometry such vectors lie in the plane $z = 0$ and intersect the plane $z = 1$ "at infinity."

3. How to make an affine senate from a projective senate of order n (with $n + 1$ members per committee)

Select any committee to be the "committee at infinity", and remove it and all its $n + 1$ members from the senate. This leaves $n(n + 1)$ committees and n^2 instructors. Committees that used to have an "instructor at infinity" in common now are parallel committees of the affine senate.

It is possible to add a committee at infinity to an affine senate, then remove a different committee and end up with a different affine senate!

Here is the small projective senate with coordinates.

Amy (0,0,1)

Bob (1,0,1)

Cal (2,0,1)

Dan (0,1,1)

Eva (1,1,1)

Fay (2,1,1)

Gus (0,2,1)

Hal (1,2,1)

Ian (2,2,1)

Jim (1,0,0) (instructor at infinity with slope 0)

Ken (1,1,0) (instructor at infinity with slope 1)

Lou (1,2,0) (instructor at infinity with slope 2)

May (0,1,0) (instructor at infinity with slope infinity)

Committees:

athletics: Amy, Bob, Cal, Jim ($y = 0$) (0, 1, 0)

budget: Dan, Eva, Fay, Jim ($y + 2z = 0$) (0, 1, 2)

compensation: Gus, Hal, Ian, Jim ($y + z = 0$) (0, 1, 1)

diversity: Amy, Dan, Gus, May ($x = 0$) (1, 0, 0)

education: Bob, Eva, Hal, May ($x + 2z = 0$) (1, 0, 2)

football: Cal, Fay, Ian, May ($x + z = 0$) (1, 0, 1)

gifts: Amy, Eva, Ian, Ken ($x + 2y = 0$) (1, 2, 0)

housing: Bob, Fay, Gus, Ken ($x + 2y + 2z = 0$) (1, 2, 2)

insurance: Cal, Dan, Hal, Ken ($x + 2y + z = 0$) (1, 2, 1)

junior faculty: Amy, Fay, Hal, Lou ($x + y = 0$) (1, 1, 0)

kitchen: Bob, Dan, Ian, Lou ($x + y + 2z = 0$) (1, 1, 2)

library: Cal, Eva, Gus, Lou ($x + y + z = 0$) (1, 1, 1)

mystery: Jim, Ken, Lou, May (the committee at infinity) ($z = 0$) (0, 0, 1)

- The vector that represents an instructor or a committee can be multiplied by 2 (equivalently, have its sign changed) and it still represents the same instructor or committee.
- The vectors that represent instructors are in V and correspond to 3×1 matrices. The vectors that represent committees are in V^* and correspond to 1×3 matrices. Adding a vector in V to a vector in V^* is nonsense!
- Plugging an instructor into the equation for a committee can be implemented as a dot product. If the result is zero, the instructor serves on the committee. All nonzero results are equivalent.

4. Here is the medium affine projective senate.

Adam $(0,0,1)$

Beth $(1,0,1)$

Carl $(u,0,1)$

Dave $(u + 1,0,1)$

Emma $(0,1,1)$

Fred $(1,1,1)$

Greg $(u,1,1)$

Herb $(u+1,1,1)$

Irma $(0,u,1)$

Jane $(1,u,1)$

Kate $(u,u,1)$

Lynn $(u + 1,u,1)$

Mike $(0,u+1,1)$

Neil $(1,u+1,1)$

Owen $(u, u + 1,1)$

Phil $(u + 1,u + 1,1)$

Quin $(1,0,0)$ (instructor at infinity with slope 0)

Ruth $(1,1,0)$ (instructor at infinity with slope 1)

Seth $(1,u,0)$ (instructor at infinity with slope u)

Tony $(1,u + 1,0)$ (instructor at infinity with slope $u + 1$)

Ursa $(0,1,0)$ (instructor at infinity with slope infinity)

athletics: Adam, Beth, Carl, Dave, Quin

budget: Emma, Fred, Greg, Herb, Quin

compensation: Irma, Jane, Kate, Lynn, Quin

diversity: Mike, Neil, Owen, Phil, Quin

education: Adam, Emma, Irma, Mike, Ursa

football: Beth, Fred, Jane, Neil, Ursa

gifts: Carl, Greg, Kate, Owen, Ursa

housing: Dave, Herb, Lynn, Phil, Ursa

insurance: Adam, Fred, Kate, Phil, Ruth

junior faculty: Beth, Emma, Lynn, Owen, Ruth

kitchen: Carl, Herb, Irma, Neil, Ruth

library: Dave, Greg, Jane, Mike, Ruth

medical: Adam, Herb, Jane, Owen, Seth

night school: Beth, Greg, Irma, Phil, Seth

overseers: Carl, Fred, Lynn, Mike, Seth

planning: Dave, Emma, Kate, Neil, Seth

quality: Adam, Greg, Lynn, Neil, Tony

recruiting: Beth, Herb, Kate, Mike, Tony

steering: Carl, Emma, Jane, Phil, Tony

tenure: Dave, Fred, Irma, Owen, Tony

unusual: Quin, Ruth, Seth, Tony, Ursa (committee at infinity)

5. Here, for reference, are the instructors in the large projective faculty senate

Alice (0,0,1)
Betty (1,0,1)
Chris (2,0,1)
Danny (3,0,1)
Emily (4,0,1)
Frank (0,1,1)
Gavin (1,1,1)
Helen (2,1,1)
Isaac (3,1,1)
James (4,1,1)
Kevin (0,2,1)
Larry (1,2,1)
Maria (2,2,1)
Nancy (3,2,1)
Oscar (4,2,1)
Patty (0,3,1)
Quint (1,3,1)
Ralph (2,3,1)
Sally (3,3,1)
Terry (4,3,1)
Uriah (0,4,1)
Viola (1,4,1)
Wally (2,4,1)
Xenia (3,4,1)
Yoric (4,4,1)
Zelda (1,0,0) (member at infinity with slope 0)
1nita (1,1,0) (member at infinity with slope 1)
2pelo (1,2,0) (member at infinity with slope 2)
3some (1,3,0) (member at infinity with slope 3)
4tune (1,4,0) (member at infinity with slope 4)
5larm (0,1,0) (member at infinity with slope infinity)

There are $5^3 = 125$ vectors in $(\mathbb{Z}_5)^3$. Remove the zero vector and there are 124. But four of these correspond to each instructor. So there are $124/4 = 31$ instructors in all.

6. Here, for reference, are the committees in the large projective faculty senate

Committees:

athletics: Alice, Betty, Chris, Danny, Emily, Zelda

budget: Frank, Gavin, Helen, Isaac, James, Zelda

compensation: Kevin, Larry, Maria, Nancy, Oscar, Zelda

diversity: Patty, Quint, Ralph, Sally, Terry, Zelda

education: Uriah, Viola, Wally, Xenia, Yoric, Zelda

football: Alice, Frank, Kevin, Patty, Uriah, 5larm

gifts: Betty, Gavin, Larry, Quint, Viola, 5larm

housing: Chris, Helen, Maria, Ralph, Wally, 5larm

insurance: Danny, Isaac, Nancy, Sally, Xenia, 5larm

junior faculty: Emily, James, Oscar, Terry, Yoric, 5larm

kitchen: Alice, James, Nancy, Ralph, Viola, 1nita

library: Betty, Frank, Oscar, Sally, Wally, 1nita

medical: Chris, Gavin, Kevin, Terry, Xenia, 1nita

night school: Danny, Helen, Larry, Patty, Yoric, 1nita

overseers: Emily, Isaac, Maria, Quint, Uriah, 1nita

planning: Alice, Helen, Oscar, Quint, Xenia, 2pelo

quality: Betty, Isaac, Kevin, Ralph, Yoric, 2pelo

recruiting: Chris, James, Larry, Sally, Uriah, 2pelo

steering: Danny, Frank, Maria, Terry, Viola, 2pelo

tenure: Emily, Gavin, Nancy, Patty, Wally, 2pelo

union: Alice, Isaac, Larry, Terry, Wally, 3some

vacation: Betty, James, Maria, Patty, Xenia, 3some

workplace: Chris, Frank, Nancy, Quint, Yoric, 3some

xenophobia: Danny, Gavin, Oscar, Ralph, Uriah, 3some

yearbook: Emily, Helen, Kevin, Sally, Viola, 3some

zoology: Alice, Gavin, Maria, Sally, Yoric. 4tune

1derful: Betty, Helen, Nancy, Terry, Uriah. 4tune

2torial: Chris, Isaac, Oscar, Patty, Viola. 4tune

3quarter: Danny, James, Kevin, Quint, Wally. 4tune

4casting: Emily, Frank, Larry, Ralph, Xenia. 4tune

5anddime:Zelda,1nita,2pelo,3some,4tune,5larm(committee at infinity)

7. Proving the geometric properties.

- Two instructors determine a unique committee.

Given one-dimensional subspaces spanned by \vec{v} and \vec{w} , form the two-dimensional subspace consisting of all vectors $a\vec{v} + b\vec{w}$. Choosing different representative vectors \vec{v} and \vec{w} for the instructors determines $a'\vec{v}' + b'\vec{w}'$, which is the same two-dimensional subspace.

A committee is represented by a vector in V^* : $\underline{\alpha} = (a, b, c)$, meaning $ax + by + cz = 0$.

If the dot product $\underline{\alpha} \cdot \vec{v} = 0$, the instructor \vec{v} satisfies the equation for the committee $\underline{\alpha}$. Since the only meaningful feature of a function value here is whether it is zero or not, multiplying either $\underline{\alpha}$ or \vec{v} by a nonzero element of F does not change anything.

Computationally, we can use the cross product notation.

Given two instructors \vec{v} and \vec{w} , the committee that they determine is $\underline{\alpha} = \vec{v} \times \vec{w}$, since $\vec{v} \times \vec{w} \cdot \vec{v} = \vec{v} \times \vec{w} \cdot \vec{w} = 0$. Multiplying \vec{v} or \vec{w} by a nonzero element of F does not change the committee.

- Two committees intersect in a unique instructor.

This is obvious geometrically in the case where $F = \mathbb{R}$: two distinct planes through the origin intersect in a line through the origin. Computationally, it looks as though we would need to solve simultaneous linear equations, but the cross product bails us out. Given a committee represented by $\underline{\alpha}$ and a committee represented by $\underline{\beta}$, the cross product $\underline{\alpha} \times \underline{\beta}$ specifies the instructor who serves on both committees, since $\underline{\alpha} \times \underline{\beta} \cdot \underline{\alpha} = \underline{\alpha} \times \underline{\beta} \cdot \underline{\beta} = 0$.

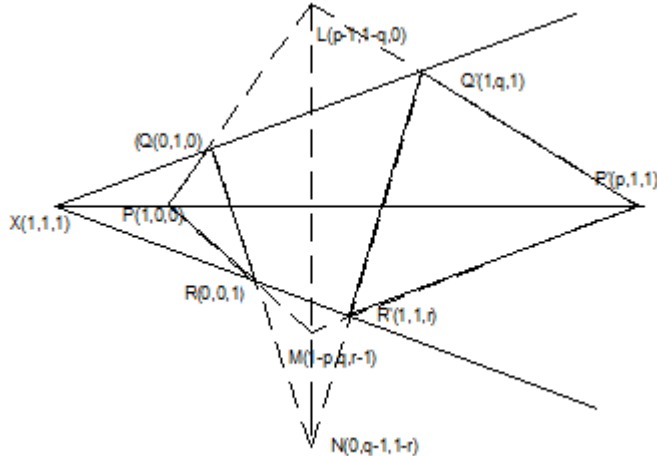
8. Introducing coordinates into a projective senate.

Let P, Q, R , and S be four instructors, no three of whom serve on the same committee. Here is how coordinates can be introduced so that

$$P = (1, 0, 0), Q = (0, 1, 0), R = (0, 0, 1), S = (1, 1, 1).$$

- Note which committee includes R and P : choose it as the x axis.
- Note which committee includes R and Q : choose it as the y axis.
- Choose the committee with P and Q as the “committee at infinity.” on which $P = (1, 0, 0), Q = (0, 1, 0)$. Remove it to get an affine senate, which still has R and S as instructors.
- Choose R as the origin for the affine senate; so its coordinates are $(0,0,1)$.
- The final step is to give S coordinates $(1,1,1)$: i.e. to introduce coordinates into the affine senate so that S has coordinates $(1, 1)$. We know how to do this by using committees that are parallel (in the affine senate) to the y -axis and to the x -axis.

9. The theorem of Desargues (most famous theorem of projective geometry)



- Choose three committees that intersect in X .
- On these committees choose points P, Q, R that are not all on the same line, then choose new points P', Q', R' , again not all on the same line.
- Find the intersection L of PQ with $P'Q'$, M of PR with $P'R'$, N of QR with $Q'R'$.
- The theorem states that L, M , and N are collinear. The proof involves nothing but cross products and is easy to do by hand if we introduce coordinates cleverly.
 - Choose $X = (1, 1, 1)$ and $P(1, 0, 0)$. Then P' is in the subspace spanned by X and P and must have coordinates of the form $(p, 1, 1)$ with $p \neq 1$.
 - Choose $Q = (0, 1, 0)$. Then Q' is in the subspace spanned by X and Q and must have coordinates of the form $(1, q, 1)$. ($q \neq 1$).
 - Choose $R(0, 1, 0)$. Then R' is in the subspace spanned by X and R and must have coordinates of the form $(1, 1, r)$. ($r \neq 1$).
 - The cross product $L = (P' \times Q') \times (P \times Q) = (1 - q, 1 - p, pq - 1) \times (0, 0, 1) = (1 - p, q - 1, 0)$.
 - Similarly, $M = (R' \times P') \times (R \times P) = (1 - r, rp - 1, 1 - p) \times (0, 1, 0) = (p - 1, 0, 1 - r)$
 - Finally, $N = (Q' \times R') \times (Q \times R) = (qr - 1, 1 - r, 1 - q) \times (1, 0, 0) = (0, 1 - q, r - 1)$.
 - The sum of the vectors for L, M , and N is 0; so they are linearly dependent. As three-component vectors they lie in a plane, which means that as projective points they are collinear.

Sample problems

1. Simple calculations in the small projective senate
 - (a) On what committee do Ken and Fay both serve?
 - (b) What instructor serves on both the insurance committee and the football committee?
 - (c) Use determinants to find whether or not Ken, Gus, and Cal serve together on any committee.

2. Introducing coordinates into the projective faculty senate

I will work with data provide by the class.

- First choose any two instructors P and Q who were not on the original “committee at infinity”.
- Next, choose an instructor R who does not serve on the same committee as P and Q .
- Finally, choose an instructor S who is not on either committee PQ or PR .

My job is to draw a diagram that will show how to assign coordinates to all 13 instructors in such a way that $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 1)$, $S = (1, 1, 1)$.

3. The theorem of Desargues

- (a) Justify the claim that if one instructor on a committee is $X = (1, 1, 1)$ and another is $P = (1, 0, 0)$, all the other instructors must be of the form $(p, 1, 1)$, where p is any element of the underlying field except 1. Confirm that it is correct for the budget committee in the large affine senate.
- (b) Draw a diagram illustrating the theorem of Desargues for the case where the committee $XMLN$ is the committee at infinity. This diagram represents a famous theorem of affine (and Euclidean) geometry.

Small group exercises.

1. Calculations in a projective senate
 - (a) Disappointed by failure of the football team to win the league championship, the dean of the small projective faculty senate decrees the football committee to be a committee at infinity, then fires all its members. List the twelve committees of the affine senate that remains, arranged into parallel pencils.
 - (b) Danny and Terry serve on one committee of the large projective senate; Larry and 2pelo serve on another. Using cross products, determine which instructor serves on both these committees. Then, using identities for dot and cross products, invent a formula involving determinants for solving this sort of problem.
 - (c) If you introduce coordinates into the medium projective senate so that Kate is $(1, 0, 0)$, Seth is $(0, 1, 0)$, Beth is $(0, 0, 1)$, and Adam is $(1, 1, 1)$, which instructor has coordinates $(1, 0, 1)$ and which has coordinates $(0, 1, 1)$?
 - (d) In the small projective senate, the dean needs to convene a meeting to discuss the possibility of improving the performance of the football team by serving more nutritious food in the dormitories. She needs a committee that has one member who serves on both athletics and football and a second member who serves on both housing and kitchen. Show how to determine this committee by calculating cross products. Then, using identities for dot and cross products, invent a formula involving determinants for solving this sort of problem.

2. The theorem of Desargues

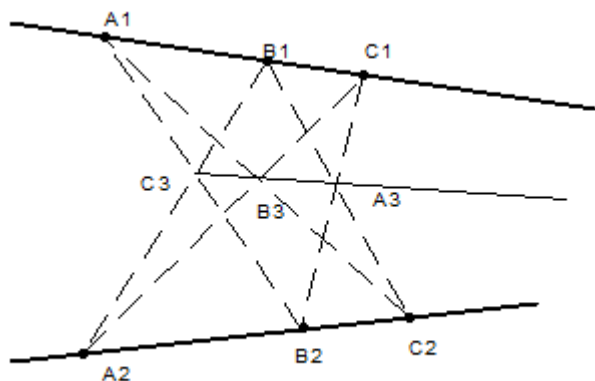
- (a) In the diagram for Desargues' theorem, use the coordinates for the large projective senate from page 6. Find a formula for the coordinates of instructor L in terms of p and q and confirm that it gives the right answer when $p = 3, q = 0$.
- (b)
- Draw a diagram representing the special case of Desargues' theorem where the committee at infinity is LMN but X is not an instructor at infinity. (This is another famous theorem of affine geometry.)
 - Draw a diagram representing the special case of Desargues' theorem where the committee at infinity includes C' and M . (This is a valid theorem of Euclidean geometry, but its proof is not obvious).
- (c) In the proof of Desargues' theorem, you can choose p and q freely, but then you must choose r so that instructors $P', Q',$ and R' are not collinear. Use determinants to find what values of r are forbidden.
- (d) The so-called "fundamental theorem of projective geometry" states that given three instructors P, Q, R who do not all serve on the same committee and any fourth instructor S different from the first three, there exists a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that maps $(1, 0, 0)$ to P , $(0, 1, 0)$ to Q , $(0, 0, 1)$ to R , and $(1, 1, 1)$ to S . Prove it by using what you know about invertibility of 3×3 matrices.

Homework problems.

1. For the “medium projective senate,” using the coordinates given on page 5, calculate cross products to answer the following questions. The answers can of course be checked by using the committee listings. In calculating, remember that $u^2 = u + 1$, $(u + 1)^2 = u$, and $u(u + 1) = 1$.
 - (a) What is the equation of the committee α on which Carl and Lynn serve? Check that Fred and Seth both satisfy this equation.
 - (b) What is the equation of the committee β on which Beth and Irma serve?
 - (c) What is the equation of the committee γ on which Dave and Neil serve?
 - (d) What instructor serves on both α and β ? On both α and γ ?
2. In the small projective senate, one might ask, “Does the instructor who serves on football and housing also serve on a committee with both Hal and Lou?”

By using identities for the dot and cross product, invent a way of solving this sort of problem by evaluating dot products only (no cross products) and check that it gives the right answer in the specified case.
3. Suppose that the proof of Desargues’ Theorem from the notes is applied to the large projective senate, with $p = 1, q = 2, r = 3$. All arithmetic is in \mathbb{Z}_5 , of course.
 - (a) Who are P', Q' , and R' ?
 - (b) Who are L, M , and N , and on what committee do they all serve?

4. The theorem of Pappus (second most famous theorem of projective geometry)



- (a) We can associate any four of the nine points in the diagram with $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,1)$ as long as no three are on a line.
 A good choice is
 $A1 = (1, 0, 0)$, $B1 = (0, 1, 0)$, $A2 = (0, 0, 1)$, $B2 = (1, 1, 1)$.
 Find a formula for $C1$ (involving p) and for $C2$ (involving q)
- (b) By calculating lots of cross products, confirm that $A3$, $B3$, and $C3$ are collinear for all values of p and q , thereby completing the proof. Use Mathematica if you wish.
- (c) Draw a diagram to illustrate the projective version of Pappus' Theorem for the case where the line $A3B3C3$ is chosen as the line at infinity.
- (d) Draw a diagram to illustrate the projective version of Pappus' Theorem for the case where the line $A1B1C1$ is chosen as the line at infinity.
- (e) Draw a diagram to illustrate the projective version of Pappus' Theorem for the case where the line $A1C3B2$ is chosen as the line at infinity.